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# Another Look at the Einstein-Maxwell Equations

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## *ABSTRACT*

An electric monopole solution to the equations of Maxwell and Einstein's general relativity is displayed. It differs from the usual one in that all components of the metric vanish at large spatial distances from the charge rather than approaching the Minkowski metric. This enables us to find an approximate solution to that for many charges.

# 1 Introduction.

A great deal of work has been done investigating the properties and symmetries [1] of the Einstein Maxwell system of equations. However, few solutions have been found to date [2]. In this paper I present a very simple static, exterior solution, in which the components of the metric vanish asymptotically. The field is that of a spherically symmetric electric charge distribution.

Such a solution is of interest as it describes a universe consisting of an electric charge and a metric for which all space time points are light like separated at large distances from the charge. The origin of this property then permits us to find an approximate solution to the case involving many charged particles.

The solution was found with the help of an algebraic computer program.

# 2 The Solution.

The equations to be solved are [3]

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi T_{\mu\nu} \quad (1)$$

$$F^{\mu\nu};_{\nu} = 4\pi J^{\mu} \quad (2)$$

where

$$T_{\mu\nu} = \frac{1}{4\pi} \left[ g^{\alpha\beta} F_{\mu\alpha} F_{\nu\beta} - \frac{1}{4} g_{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} \right], \quad (3)$$

$R_{\mu\nu}$  is the contracted curvature (Ricci) tensor,  $R$  the scalar curvature,  $g_{\mu\nu}$  the metric tensor,  $F_{\mu\nu}$  the electromagnetic field strength, and  $T_{\mu\nu}$  the Maxwell stress-energy tensor.

Eq.(2) may be written as

$$\frac{1}{\sqrt{-g}} (\sqrt{-g} F^{\mu\nu})_{;\nu} = 4\pi J^{\mu} \quad (4)$$

where  $g$  is the determinant of the metric tensor.

The metric is conformally related to that of flat space so that it is given by the line element

$$ds^2 = h^2(dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 - dt^2). \quad (5)$$

The field strength is given by

$$F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu} \quad (6)$$

where

$$A_0 = \frac{e}{r} \quad (7)$$

for  $r > r_0$  and  $A_i = 0$  for  $i = 1, 2, 3$ .

A static monopole solution to these equations for  $r > r_0$  is

$$h = \frac{d}{r} \quad (8)$$

with

$$d = \frac{e\sqrt{G}}{c^2} \quad (9)$$

where  $d$  and  $e$  are constants, and  $G$  and  $c$  have been temporarily restored for the purposes of clarity [4].

$J^\mu$  is that of a static, spherically symmetric, electric charge distribution contained inside the region  $r \leq r_0$  and of total charge  $e$ . This can be verified by assuming the existence of an interior solution that can be joined to this exterior one. One then integrates Eq.(4) over a spatial volume enclosing the charge, using appropriate factors of  $\sqrt{-g}$ , *i.e.*,

$$q = \int_{x^0=const.} \sqrt{-g} \times \frac{1}{4\pi\sqrt{-g}} (\sqrt{-g}F^{0i})_{,i} dr d\theta d\phi. \quad (10)$$

Since, the integrand has only a  $\sin \theta$  dependence on angle, the integral reduces to the value of  $\sqrt{-g}F^{0i}/\sin \theta$  evaluated at  $r$  ( $r > r_0$ ), whereupon Eqs.(5) and (7) are then used to find  $q = e$ . (I have assumed that  $\sqrt{-g}F^{0i}/\sin \theta = 0$  at  $r = 0$  which will be true except for the point charge where it equals  $e$ . However, the point charge is not considered in this paper since it is not a solution to the integrated form of Eq.(1).)

It is interesting to note that an electrically neutral test particle at rest in this universe will have an energy  $E = m\sqrt{-g_{00}} = mh$ . Hence  $E$  will vanish at large spatial coordinate distances from the charge.

### 3 Comparison to the Reissner-Nordström Solution.

The Reissner-Nordström solution [5] is the spherically symmetric, static, exterior field of a charged distribution of mass. It is given by

$$ds^2 = \frac{dr^2}{l} + r^2(d\theta^2 + \sin^2 \theta d\phi^2) - l dt^2 \quad (11)$$

where

$$l = 1 - \frac{2M}{r} + \frac{e^2}{4\pi r^2}. \quad (12)$$

If  $M$  is set to zero we can then make a direct comparison between Eqs.(5) and (11). In Eq.(5) all the components of the metric vanish for large distances from the charge, so that  $ds^2 \rightarrow 0$  while in Eq.(11)  $ds^2$  approaches the Minkowski expression for flat space.

On the other hand as  $r \rightarrow r_0$  and  $r_0 \rightarrow 0$ , Eq.(5) gives  $ds^2 \rightarrow (d/r)^2[dr^2 - dt^2]$  which can be light-like, space-like or time-like while Eq.(11) gives  $ds^2 \rightarrow -(e^2/4\pi r^2)dt^2$ .

We also note that a test particle at rest in the universe of the metric of Eq.(11) will have an energy  $E = m\sqrt{1 + e^2/4\pi r^2}$  and hence approaches  $m$  at large distances from the charge in contrast to the energy of the test particle in section 2 which vanished at large distances. However, the energies have the same behavior at very small distances.

## 4 An Approximate Solution for Many Charges

Referring to Eq.(5) we note that  $ds^2$  goes like  $d^2/r^2$ . Let us now consider charges of the order of magnitude of that of the electron. Referring to the values given in Ref.[4], we see that if we are at atomic distances, *e.g.*, the Bohr radius, from the charge,  $ds^2$  will have dropped by a factor of  $d^2/r^2 \approx 10^{-52}$  from what it was in the neighborhood of the charge. Thus if we have a gas of these charges, separated by atomic distances, the metric will either be effectively zero if we are away from any charges, or will be given by Eq.(5) (with the origin shifted to the location of the  $i^{th}$  charge) if we are within a distance  $d$  of that charge. Hence, in this approximation

$$ds^2 = \sum_i h_i^2(dx^2 + dy^2 + dz^2 - dt^2) \quad (13)$$

where

$$h_i = \frac{d}{|\mathbf{r} - \mathbf{r}_i|} \quad (14)$$

## 5 The Computer Program

I constructed a symbolic computer program to help find this solution. It was a bit like using a canon to shoot down a pea. It was written in both Mathematica and MapleV.

The program takes as input, the coordinates, the covariant components of the metric and the covariant components of the electromagnetic vector potential. It then outputs the determinant of the metric, the contravariant components of the metric, the electromagnetic field strengths, the current density, the Maxwell stress-energy tensor, the Christoffel symbols, the curvature tensor, the Ricci tensor, the scalar curvature, the Einstein tensor and the Einstein equations of motion. If the last is not satisfied, it gives the remaining terms by which it fails. I will restrict the following detailed discussion to MapleV.

In order to distinguish between the covariant and the contravariant components of the metric, I used  $gl[i,j]$  for the covariant and  $gu[i,j]$  for the contravariant components. The program itself very closely follows the equations one would normally write down while doing a calculation. As an example I give the lines for computing the Christoffel symbols.

```
gam:=array(1..4,1..4,1..4):
for i to 4 do for j to 4 do for k to 4 do
gam[i,j,k]:=1/2*simplify( sum( 'gu[i,l]*(diff( gl[l,j], x[k]) + diff( gl[l,k], x[j])
- diff( gl[j,k], x[l]))',l=1..4)) od: od: od: (15)
```

The input is accomplished by typing the data into a first program rather than interacting with the computer directly. This is then called by a second program. This system enables the input program to be short enough so that one can save a number of them efficiently. Results one wishes to save are specified in the second program and may be used as input for further work.

## 6 Summary

In sections 2 and 3, we have two very different solutions to Eqs.(1) and (2) for a point charge. Eq.(11) is the well known solution which asymptotically approaches flat space while for Eq.(5)  $ds^2 \rightarrow 0$  asymptotically. In section 4 we then discussed

how concentration of the metric about a charge to within distances tiny compared to atomic sizes enables one to obtain an approximate solution for the case of many charges separated by distances of atomic dimensions.

Of course, quantum mechanical field theoretic considerations will play a dominant role at the small distances discussed in this paper. However, a model such as the one presented here might serve as the background field in a quantization procedure.

While the computer program is not a substitute for keen insight, nor even pen and paper, it should be helpful in looking at more complex solutions to the Einstein-Maxwell problem.

## References

- [1] C.Hoenselaers, William Kinnersley, and Basilis C. Xanthopoulos J. Math. Phys. **20**, 2530 (1979) and other references given therein.
- [2] Y. Srivstava and A. Widom, *Phys. Letts.* **B280**, 52 (1992)
- [3] I use units  $G = c = 1$ .
- [4] We note that if  $e = 4.8 \times 10^{-10}$  esu (the electron charge), then  $d = 1.4 \times 10^{-34}$  cm.
- [5] H. Reissner, *Ann. Phys.* **50**, 106 (1916)